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ON PROPERTIES OF THE APPROXIMATE PEANO DERIVATIVES(1)

BY

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ABSTRACT. The notion of kth approximate Peano differentiation not only generalizes kth ordinary differentiation but also kth Peano differentiation and kth L_p differentiation. Recently, M. Evans has shown that a kth approximate Peano derivative at least shares with these other derivatives the property of belonging to Baire class one. In this paper the author extends the properties possessed by a kth approximate Peano derivative by showing that it is like the above derivatives in that it also possesses the following properties: Darboux, Denjoy, Zahorski, and a new property stronger than the Zahorski property, Property Z.

1. **Introduction.** Let k be a positive integer. Let f be a real-valued, measurable function defined on the closed interval $I = [a, b]$ and let $x \in I$. If there are numbers $f_{(1)}(x), f_{(2)}(x), \ldots, f_{(k)}(x)$ and a measurable set E having 0 as a point of density so that

$$f(x+h) = f(x) + hf_{(1)}(x) + \ldots + (h^k/k!)f_{(k)}(x) + o(h^k)$$

as $h \longrightarrow 0$, $h \in E$ and $x + h \in I$, then the number $f_{(k)}(x)$ is called the kth approximate Peano derivative of f at x. We will find it convenient to write $f_{(0)}(x) = f(x)$. It is easily seen from the definition that if $f_{(k)}(x)$ exists then so does $f_{(n)}(x)$ for $0 \leq n < k$. Also, $f_{(1)}(x) = f'_{\text{ap}}(x)$, the approximate derivative. The notion of kth approximate Peano differentiation not only generalizes kth ordinary differentiation but also kth Peano differentiation and kth L_p differentiation. For definitions of the latter two types of derivatives see [2].

Recently, Evans [2] proved that if $f_{(k)}$ is defined on I then $f_{(k)}$ is in the first class of Baire (a pointwise limit of continuous functions). The purpose of this paper is to show that $f_{(k)}$ also has the following properties: (1) Darboux,

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(1) The results in this paper were part of a dissertation presented to Michigan State University for the degree of Doctor of Philosophy and written under the direction of Professor Clifford E. Weil.

(2) Denjoy, (3) Zahorski, and (4) Property Z. That an ordinary derivative, approximate derivative, kth Peano derivative and kth L_p derivative are in the first class of Baire and possess the above four properties, we refer the reader to [1]–[4], [6]–[11].

We begin in §2 by defining the four properties stated above and by giving notation and terminology which will be used throughout this paper. In §3, a density lemma is proved which plays a key role in §4 where the following major result is proved. If $f_{(k)}$ is defined on I and if $f_{(k)}$ is bounded above or below on I then $f^{(k)}$, the ordinary kth derivative of f, exists and $f^{(k)} = f_{(k)}$ on I. Properties 1 and 2 are shown to hold for $f_{(k)}$ in §5 by using known theorems together with the major result. In §6, the final section, a lemma is proved from which property 4 is shown to hold for $f_{(k)}$. Again from a known theorem, property 3 is shown to follow from property 4 for $f_{(k)}$.

2. **Notation, terminology and definitions.** All of the functions in this paper are assumed to be real-valued, measurable functions defined on the closed interval $I = [a, b]$ unless specified otherwise. R will denote the real numbers and if $E \subset R$ is a measurable set then we denote the measure of E by either $m(E)$ or $|E|$. The notation $E\text{-}\lim_{y\to x}$ denotes $\lim_{y\to x, y\in E}$. For convenience we now define the four properties stated in the introduction.

Let g be a function defined on I.

1. g possesses the Darboux property if g maps connected sets of I into connected sets.

2. g satisfies the Denjoy property if, for every open interval (c, d), $g^{-1}((c, d))$ either is empty or has positive measure.

3. g has the Zahorski property if the following condition is fulfilled: If $c < d$, $x \in g^{-1}((c, d))$, and if $\{I_n\}$ is a sequence of closed intervals of I not containing x so that $I_n \longrightarrow x$ and $m(I_n \cap g^{-1}((c, d))) = 0$ for all n, then

$$\lim_{n\to\infty} \frac{m(I_n)}{\operatorname{dist}(x, I_n)} = 0.$$

The notation $I_n \longrightarrow x$ means that every neighborhood of x contains all but finitely many of the I_n's.

4. g is said to have Property Z if the following condition is satisfied: If for each $\epsilon > 0$ and each sequence $\{I_n\}$ of closed intervals of I such that $I_n \longrightarrow x$ and $g(y) \geqslant g(x)$ on I_n or $g(y) \leqslant g(x)$ on I_n for each n, then

$$\lim_{n\to\infty} \frac{m\{y \in I_n : |g(y) - g(x)| \geqslant \epsilon\}}{m(I_n) + \operatorname{dist}(x, I_n)} = 0.$$

Let $E \subset R$ be a measurable set and let $x \in R$. Define

$$d(x, E) = \lim_{h \to 0} \frac{m(E \cap [x - h, x + h])}{2h},$$

$$d_+(x, E) = \lim_{h \to 0^+} \frac{m(E \cap [x, x + h])}{h},$$

and $d_-(x, E)$ in the obvious fashion. If $d(x, E) = 1$ (0) then x is called a point of density (dispersion) of E. If $d_+(x, E) = 1$ (0) then x is called a point of right-hand density (dispersion) of E; if $d_-(x, E) = 1$ (0) then x is said to be a point of left-hand density (dispersion) of E.

The following simple observations, which will be used later, are now noted. If $d_+(0, E) = d_+(0, F) = 0$ (1) then $d_+(0, E \cap F) = 0$ (1) and $d_+(0, E^c) = 1$ (0), where E^c is the complement of E.

REMARK. If $x = a$ $(x = b)$ in the definition of a kth approximate Peano derivative then the expression "there exists a measurable set E having 0 as a point of density" is understood to mean that $E \subset [0, \infty)$ $(E \subset (-\infty, 0])$ and 0 is a point of right-hand (left-hand) density of E.

3. **A density lemma.** If $E \subset R$ and $\lambda \in R$ then we define $\lambda E = \{\lambda e : e \in E\}$. Before proceeding to the density lemma we need

LEMMA 3.1. *Let $E \subset R$ be a set of finite measure and let $\epsilon > 0$. Then there exists a $\delta > 0$ such that $|\lambda E - E| < \epsilon$, whenever $|1 - \lambda| < \delta$.*

PROOF. If $|E| = 0$ or if E is a finite union of intervals then the lemma is easily verified. Thus, assume E is a set of finite measure and let G be an open set such that $E \subset G$ and $|G - E| < \epsilon/3$. Without loss of generality we may assume $G = \bigcup_{n=1}^{\infty} I_n$, a pairwise disjoint union of open intervals. Choose $N > 0$ so that if $H = \bigcup_{n=N+1}^{\infty} I_n$, then $|H| < \epsilon/6$. Choose $0 < \delta < 1$ so that $|\lambda F - F| < \epsilon/3$, whenever $|1 - \lambda| < \delta$, where $F = \bigcup_{n=1}^{N} I_n$. If $|1 - \lambda| < \delta$, then

$$\begin{aligned} |\lambda E - E| &\leq |\lambda G - E| \leq |\lambda G - G| + |G - E| \leq |\lambda G - F| + |G - E| \\ &\leq |(\lambda F \cup \lambda H) - F| + |G - E| \leq |\lambda F - F| + \lambda |H| + |G - E| \\ &\leq \epsilon/3 + 2 \cdot \epsilon/6 + \epsilon/3 = \epsilon. \end{aligned}$$

DENSITY LEMMA 3.2. *Let $d_+(0, E) = 1$. Then there are numbers α_n, β_n such that $0 < \alpha_n < \alpha_{n+1} < 1 < \beta_{n+1} < \beta_n$ $(n = 1, 2, \ldots)$, $\alpha_n \longrightarrow 1, \beta_n \longrightarrow 1$ and*

$$d_+\left(0, \bigcap_{n=1}^{\infty} \alpha_n E\right) = d_+\left(0, \bigcap_{n=1}^{\infty} \beta_n E\right) = 1.$$

PROOF. Set $H = E^c$, then $d_+(0, H) = 0$. There are numbers δ_k such that $1/k > \delta_k > \delta_{k+1} > 0$ and $|H \cap (0, t)| < t/k^2$ for each $t \in (0, 2\delta_k)$ $(k = 1, 2, \ldots)$. Set $F = H \cap (0, 2)$. By Lemma 3.1 there are numbers α_n, β_n such that

$$n/(n+1) < \alpha_n < \alpha_{n+1} < 1 < \beta_{n+1} < \beta_n < (n+1)/n$$

and

$$|\alpha_n F - F| < \delta_n/2^n, \qquad |\beta_n F - F| < \delta_n/2^n$$

for $n = 1, 2, \ldots$. Let $\epsilon > 0$; let k be an integer greater than $3/\epsilon$ and let $0 < h < \delta_k$. Choose $j \geq k$ so that $\delta_{j+1} \leq h < \delta_j$. Since $\alpha_n H \cap (0, h) = \alpha_n(H \cap (0, h/\alpha_n))$ and since $h/\alpha_n < 2h < 2\delta_j < 2$, we have

$$|\alpha_n H \cap (0, h)| = \alpha_n |H \cap (0, h/\alpha_n)| < h/j^2$$

and

$$\alpha_n H \cap (0, h) \subset \alpha_n F$$

for each n. As

$$\bigcup_{n=1}^{\infty} \alpha_n H \subset \left(\bigcup_{n=1}^{j} \alpha_n H\right) \cup \left(\bigcup_{n=j+1}^{\infty} (\alpha_n H - H)\right) \cup H$$

and $(\alpha_n H - H) \cap (0, h) \subset \alpha_n F - F$, we get

$$\begin{aligned}\left|\left(\bigcup_{n=1}^{\infty} \alpha_n H\right) \cap (0, h)\right| &\leq \sum_{n=1}^{j} |\alpha_n H \cap (0, h)| \\ &\quad + \sum_{n=j+1}^{\infty} |\alpha_n F - F| + |H \cap (0, h)| \\ &< j \cdot h/j^2 + \sum_{n=j+1}^{\infty} \delta_n/2^n + h/j^2 \\ &< h/j + h/2^j + h/j^2 \\ &< 3h/j \leq 3h/k < \epsilon h.\end{aligned}$$

Thus $d_+(0, \bigcup_{n=1}^{\infty} \alpha_n H) = 0$. Since

$$\left(\bigcup_{n=1}^{\infty} \alpha_n H\right)^c = \bigcap_{n=1}^{\infty} (\alpha_n H)^c = \bigcap_{n=1}^{\infty} \alpha_n H^c = \bigcap_{n=1}^{\infty} \alpha_n E,$$

$d_+(0, \bigcap_{n=1}^{\infty} \alpha_n E) = 1$. Similarly it can be proved that $d_+(0, \bigcap_{n=1}^{\infty} \beta_n E) = 1$.

4. **The major theorem.** In this section we deduce the fundamental result stated in

THEOREM 4.1. *Let $f_{(k)}$ be defined on I.*

(i) *If $f_{(k)} > 0$ on I, then $f_{(k-1)}$ is continuous and nondecreasing on I.*

(ii) *If $f_{(k)}$ is bounded above or below on I, then $f_{(k)} = f^{(k)}$ on I.*

The proof of this theorem will require some additional lemmas.

LEMMA 4.2. *Let $f_{(k)}$ be defined on $I = [a, b]$. Assume $f_{(1)}$ is nondecreasing on I, and if $k \geq 2$ furthermore assume $f_{(2)}(a) = f_{(3)}(a) = \ldots = f_{(k-1)}(a) = 0$. Then $(f_{(1)})_{(k-1)}(a) = f_{(k)}(a)$.*

PROOF. By subtracting from f a multiple of x, we may assume that $f_{(1)}(a) = 0$. By hypothesis there exists a measurable set F such that $d_+(0, F) = 1$ and

$$F\text{-}\lim_{h\to 0} \frac{1}{h^k} \{f(a+h) - f(a) - Ah^k\} = 0, \tag{4.3}$$

where $A = f_{(k)}(a)/k!$.

By the Density Lemma there exist two sequences of positive real numbers $\{\alpha_n\}$ and $\{\beta_m\}$ such that $\alpha_n \longrightarrow 0$, $\beta_m \longrightarrow 0$ and

$$d_+\left(0, \bigcap_{n=1}^{\infty} (1-\alpha_n)F\right) = d_+\left(0, \bigcap_{m=1}^{\infty} (1+\beta_m)F\right) = 1.$$

If we set

$$E = F \cap \left[\bigcap_{n=1}^{\infty} (1-\alpha_n)F\right] \cap \left[\bigcap_{m=1}^{\infty} (1+\beta_m)F\right],$$

then $d_+(0, E) = 1$. To complete the proof of the lemma we need only show

$$E\text{-}\lim_{h\to 0} \frac{f_{(1)}(a+h)}{h^{k-1}} = Ak.$$

Let $\epsilon > 0$ be given; choose n, m so that if $\alpha = \alpha_n/(1-\alpha_n)$ and $\beta = \beta_m/(1+\beta_m)$ then

$$A \cdot \frac{(1+\alpha)^k - 1}{\alpha} < Ak + \frac{\epsilon}{2} \quad \text{and} \quad A \cdot \frac{1-(1-\beta)^k}{\beta} > Ak - \frac{\epsilon}{2}.$$

Set

$$\epsilon' = \min\left(\frac{\epsilon\alpha}{2[(1+\alpha)^k+1]}, \frac{\epsilon\beta}{2[(1+\beta)^k+1]}\right).$$

By (4.3) there exists a $\delta' > 0$ such that $|f(a+h) - f(a) - Ah^k| < \epsilon' h^k$ whenever $0 < h < \delta'$, $h \in F$. If $0 < u < v < \delta'$ and $u, v \in F$ then

$$|[f(a+v) - f(a+u)] - A(v^k - u^k)| < \epsilon'(v^k + u^k).$$

Hence,

$$A\frac{(v^k - u^k)}{v-u} - \epsilon'\frac{(v^k+u^k)}{v-u} < \frac{f(a+v) - f(a+u)}{v-u} < A\frac{(v^k-u^k)}{v-u} + \epsilon'\frac{(v^k+u^k)}{v-u}.$$

Since $f_{(1)}$ is nondecreasing on $[a, b]$ and $f_{(1)} = f'_{\mathrm{ap}}$ we have $f_{(1)} = f'$ on $[a, b]$ (see [3]) and hence

$$f_{(1)}(a+u) \leqslant \frac{f(a+v) - f(a+u)}{v-u} \leqslant f_{(1)}(a+v).$$

Thus, whenever $0 < u < v < \delta'$ and $u, v \in F$,

$$f_{(1)}(a+u) < A\frac{(v^k-u^k)}{v-u} + \epsilon'\frac{(v^k+u^k)}{v-u} \tag{4.4}$$

and

$$f_{(1)}(a+v) > A\frac{(v^k-u^k)}{v-u} - \epsilon'\frac{(v^k+u^k)}{v-u}. \tag{4.5}$$

Set $\delta = \min\{\delta'/(1+\alpha), \delta'(1-\beta)\}$ and let $h \in E$ such that $0 < h < \delta$. Since $h \in (1-\alpha_n)F$, there exists a $v \in F$ such that $h = (1-\alpha_n)v$. Hence, $v = \{1 + [\alpha_n/(1-\alpha_n)]\}h = (1+\alpha)h$ and $h < v < \delta'$. Thus from (4.4) we have

$$\begin{aligned} \frac{f_{(1)}(a+h)}{h^{k-1}} &< A\frac{[h^k(1+\alpha)^k - h^k]}{\alpha h^k} + \epsilon'\frac{[h^k(1+\alpha)^k + h^k]}{\alpha h^k} \\ &< A[(1+\alpha)^k - 1]/\alpha + \epsilon'[(1+\alpha)^k + 1]/\alpha \\ &< Ak + \epsilon/2 + \epsilon/2 < Ak + \epsilon. \end{aligned} \tag{4.6}$$

Moreover, since $h \in (1+\beta_m)F$ there exists a $u \in F$ such that $h = (1+\beta_m)u$. Hence, $u = \{1 - [\beta_m/(1+\beta_m)]\}h = (1-\beta)h$ and $u < h < \delta'$. Thus from (4.5) it follows that

$$f_{(1)}(a+h)/h^{k-1} > Ak - \epsilon. \tag{4.7}$$

Thus from (4.6) and (4.7) we have

$$|f_{(1)}(a+h)/h^{k-1} - Ak| < \epsilon$$

whenever $0 < h < \delta$ and $h \in E$.

COROLLARY 4.8. *Let $f_{(k)}$ be defined on $I = [a, b]$. Assume $f_{(1)}$ is nondecreasing on I, and if $k \geq 2$ furthermore assume $f_{(2)}(b) = f_{(3)}(b) = \ldots = f_{(k-1)}(b) = 0$. Then $(f_{(1)})_{(k-1)}(b) = f_{(k)}(b)$.*

PROOF. Define a function g on $[-b, -a]$ as follows:

$$g(x) = f(-x) \quad \text{for each } x \in [-b, -a].$$

Then $g_{(k)}(x)$ exists for each $x \in [-b, -a]$ and $g_{(n)}(x) = (-1)^n f_{(n)}(-x)$ for $n = 0, 1, \ldots, k$. It is easy to verify that $g_{(1)}$ is nondecreasing on $[-b, -a]$, and that if $k \geq 2$, $g_{(2)}(-b) = g_{(3)}(-b) = \ldots = g_{(k-1)}(-b) = 0$. The proof of the corollary is now easily completed by applying Lemma 4.2.

COROLLARY 4.9. *Let $f_{(2)}$ be defined on I. If $f_{(1)}$ is nondecreasing on I, then $(f_{(1)})_{(1)} = f_{(2)}$ on I.*

LEMMA 4.10. *Suppose f has $(k-1)$ derivatives at the point x, then for each sufficiently small nonzero h, there is a $0 < \theta < 1$ such that*

$$\frac{(k-2)!}{h^{k-2}}\left\{f(x+h) - \sum_{n=0}^{k-1}\frac{h^n}{n!}f^{(n)}(x)\right\}$$
$$= f^{(k-2)}(x+\theta h) - f^{(k-2)}(x) - \theta h f^{(k-1)}(x)$$

where $f^{(0)}(x) = f(x)$.

PROOF. Let

$$g(t) = f(x+t) - \sum_{n=0}^{k-1}\frac{t^n}{n!}f^{(n)}(x). \tag{4.11}$$

Then g is $(k-2)$ times differentiable around 0 and

$$g^{(j)}(t) = f^{(j)}(x+t) - \sum_{n=0}^{k-j-1}\frac{t^n}{n!}f^{(n+j)}(x) \tag{4.12}$$

for $j = 0, 1, \ldots, (k-2)$. By the extended mean value theorem for each sufficiently small h there exists a $0 < \theta < 1$ so that

$$g(h) = \sum_{n=0}^{k-3}\frac{h^n}{n!}g^{(n)}(0) + \frac{h^{k-2}}{(k-2)!}g^{(k-2)}(\theta h)$$

where $g^{(0)}(0) = g(0)$. By (4.12) it follows that $g^{(j)}(0) = 0$ for $j = 0, 1, \ldots, (k-3)$; hence

$$g(h) = \frac{h^{k-2}}{(k-2)!} g^{(k-2)}(\theta h). \tag{4.13}$$

Thus, by replacing the left-hand side of (4.11) by (4.13) we have

$$f(x+h) - \sum_{n=0}^{k-1} \frac{h^n}{n!} f^{(n)}(x) = \frac{h^{k-2}}{(k-2)!} g^{(k-2)}(\theta h).$$

If $h \neq 0$ then this last equation together with (4.12) yields

$$\begin{aligned} \frac{(k-2)!}{h^{k-2}} \left\{ f(x+h) - \sum_{n=0}^{k-1} \frac{h^n}{n!} f^{(n)}(x) \right\} &= g^{(k-2)}(\theta h) \\ &= f^{(k-2)}(x+\theta h) - \sum_{n=0}^{1} \frac{(\theta h)^n}{n!} f^{(k+n-2)}(x) \\ &= f^{(k-2)}(x+\theta h) - f^{(k-2)}(x) - \theta h f^{(k-1)}(x). \end{aligned}$$

Before stating the next two lemmas, proofs of which can be found in the papers of Verblunsky [8] and Zygmund [12] respectively, we need the following definitions.

DEFINITION 4.14. A function f defined on an interval is said to be convex if for every pair of points P_1, P_2 on the curve $y = f(x)$ the points of the arc P_1P_2 are below, or on, the chord P_1P_2.

DEFINITION 4.15. Let f be a function defined in a neighborhood of x. Then define

$$\overline{D}_2 f(x) = \limsup_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}.$$

$\overline{D}_2 f(x)$ is called the upper symmetric second derivative of f at x.

REMARK. It can easily be shown that if $f''(x)$ exists at x then $\overline{D}_2 f(x) = f''(x)$. However, the upper symmetric second derivative may exist at a point without the second derivative existing.

LEMMA 4.16. *Let f have a finite derivative at each point of (a, b). Suppose that for each $x_0 \in (a, b)$ there are, in every neighborhood of $(x_0, f(x_0))$, points of the graph of f above the line $y = f(x_0) + f'(x_0)(x - x_0)$. Then f is convex on (a, b).*

LEMMA 4.17. *A necessary and sufficient condition for a continuous function f to be convex on (a, b) is that $\overline{D}_2 f(x) \geq 0$ for each x in (a, b).*

The following lemma is a special case of Lemma 2 in [2].

LEMMA 4.18. *Suppose $f_{(2)}$ exists at a point $x \in (a, b)$. Then there exists a measurable set E so that $d(0, E) = 1$ and*

$$E\text{-}\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f_{(2)}(x).$$

COROLLARY 4.19. *Suppose $f_{(2)}$ exists on (a, b) and $f_{(2)} \geq 0$ on (a, b). Then $\overline{D}_2 f \geq 0$ on (a, b).*

In what follows we shall use without specific reference several well-known results. We list these results here without proof.

Let g be a function defined on an interval J and let g have an ordinary derivative g' on J. If g is convex on J then g' is nondecreasing on J.

Let g be a function defined on $[a, b]$. If g is nondecreasing on (a, b) and has the Darboux property on $[a, b]$ then g is nondecreasing on $[a, b]$.

Let g be a function defined on an interval J. If g is nondecreasing on J and has the Darboux property on J, then g is continuous on J.

Let g be a function of Baire class one on $[a, b]$. Then every nonempty closed set F, contained in $[a, b]$, contains points of continuity of g relative to F.

Let g be a function defined on an interval J and assume g'_{ap} exists at each point in J. Then the following are true (see [3]):

(1) g'_{ap} is a function of Baire class one on J,

(2) g'_{ap} has the Darboux property on J,

(3) if g'_{ap} is bounded above or below on J then $g'_{\text{ap}} = g'$ on J.

LEMMA 4.20. *Let f be a function satisfying the following two conditions on $[a, b]$:*

(i) *f'_{ap} exists for each x in $[a, b]$;*

(ii) *$\overline{D}_2 f \geq 0$ on (a, b).*

Then f'_{ap} is continuous and nondecreasing on $[a, b]$.

PROOF. Let G be the set of all points x in $[a, b]$ with the property that there is a neighborhood of x on which f'_{ap} is bounded. Then G is an open set. Let $(c, d) \subset G$; then a simple compactness argument shows f'_{ap} is bounded on $[c', d']$, where $c < c' < d' < d$. Hence $f'_{\text{ap}} = f'$ on $[c', d']$. Therefore it follows that $f'_{\text{ap}} = f'$ on (c, d). Since f is continuous on (c, d) and $\overline{D}_2 f \geq 0$ on (c, d), f is convex on (c, d) by Lemma 4.17. Hence f'_{ap} is nondecreasing on

(c, d). Moreover, since f'_{ap} has the Darboux property on $[c, d]$ it follows that f'_{ap} is continuous and nondecreasing on $[c, d]$. In particular, f'_{ap} is continuous and nondecreasing in the closure of each component of G.

To complete the proof of the lemma we show $G = [a, b]$. Let $H = [a, b] - G$; then H is a closed set having no isolated points. Suppose H is nonempty. Then H is a perfect set. Since f'_{ap} is a function of Baire class one on $[a, b]$ there exists an $x_0 \in H$ so that f'_{ap} is continuous at x_0 relative to H. Hence there is an $M \geqslant 0$ and a $\delta > 0$ such that $|f'_{\text{ap}}(x)| \leqslant M$ for each $x \in [x_0 - \delta, x_0 + \delta] \cap H$. Set

$$c = \min\{x\colon x \in [x_0 - \delta, x_0] \cap H\}, \quad d = \max\{x\colon x \in [x_0, x_0 + \delta] \cap H\}.$$

Notice that since H is perfect, $c, d \in H$ and $c < d$. If $x \in (c, d) - H$ then there exists a component of G, say (α, β), where $\alpha, \beta \in H$ such that $x \in (\alpha, \beta) \subset (c, d)$. From the first part of the proof f'_{ap} is nondecreasing on $[\alpha, \beta]$; hence

$$-M \leqslant f'_{\text{ap}}(\alpha) \leqslant f'_{\text{ap}}(x) \leqslant f'_{\text{ap}}(\beta) \leqslant M.$$

Thus, for each $x \in (c, d)$, $|f'_{\text{ap}}(x)| \leqslant M$ and so $(c, d) \subset G$.

Since $x_0 \in H$, $x_0 \notin (c, d)$; so either $x_0 = c$ or $x_0 = d$. But if $x_0 = c$ then $(x_0 - \delta, x_0) \subset G$ and there exists a number $M' \geqslant 0$, so that f'_{ap} is bounded by M' on $[x_0 - \delta, x_0]$. In the last paragraph it was shown that f'_{ap} was bounded by M on (x_0, d). So f'_{ap} is bounded by $\max\{M, M'\}$ on $(x_0 - \delta, d)$ and $x_0 \in G$. Similarly, it can be shown that if $x_0 = d$ then $x_0 \in G$. Thus, the assumption that $H \neq \emptyset$ is false. Therefore, $H = \emptyset$ and $G = [a, b]$.

PROOF OF THEOREM 4.1. Consider first the case $k = 1$. If $f_{(1)} > 0$ on $[a, b]$ then $f_{(1)} = f'$ on $[a, b]$. Thus, $f_{(0)} = f$ is continuous and nondecreasing on $[a, b]$. Moreover, if $f_{(1)}$ is bounded either above or below on $[a, b]$ then again $f_{(1)} = f'$ on $[a, b]$. Thus, the theorem holds when $k = 1$.

Secondly, consider $k = 2$. By Corollary 4.19 and Lemma 4.20 the proof of (i) is immediate. Turning to case (ii), there is no loss of generality to assume $f_{(2)} > 0$ on $[a, b]$. From (i) it follows that $f_{(1)}$ is nondecreasing on $[a, b]$; hence $f_{(1)} = f'$ on $[a, b]$. By Corollary 4.9, $(f')_{(1)} = f_{(2)}$ on $[a, b]$. Moreover, by assumption $(f')_{(1)} > 0$ on $[a, b]$; hence $(f')_{(1)} = (f')' = f^{(2)}$. Thus, $f_{(2)} = f^{(2)}$ on $[a, b]$.

We may now assume that $k > 2$, and we can complete the proof by induction. We therefore assume the following:

If f possesses a $(k-1)$th approximate Peano derivative everywhere on an interval $[a, b]$, then for $1 \leqslant n \leqslant (k-1)$:

(i) if $f_{(n)} > 0$ on $[a, b]$, then $f_{(n-1)}$ is continuous and nondecreasing on $[a, b]$;

(ii) if $f_{(n)}$ is bounded either above or below on $[a, b]$, then $f_{(n)} = f^{(n)}$ on $[a, b]$.

Let $k > 2$ and assume $f_{(k)} > 0$ at each point in $[a, b]$. Let G be the set of all points x of $[a, b]$ with the property that there is a neighborhood of x on which $f_{(k-1)}$ is bounded. Obviously G is open. Let $(c, d) \subset G$. If $c < \alpha < \beta < d$, then a simple compactness argument shows $f_{(k-1)}$ is bounded on $[\alpha, \beta]$. By (ii) of the induction hypothesis, $f_{(k-1)} = f^{(k-1)}$ on $[\alpha, \beta]$ and therefore $f_{(k-2)} = f^{(k-2)}$ on $[\alpha, \beta]$. Moreover, these relations hold on (c, d). Thus, $f_{(k-1)} = f'_{(k-2)}$ on (c, d) and $f_{(k-2)}$ is continuous on (c, d). If $x \in (c, d)$ then there exists a measurable set E such that 0 is a point of density of E and

$$f(x+h) = \sum_{n=0}^{k-1} \frac{h^n}{n!} f_{(n)}(x) + \frac{h^k}{k!}[f_{(k)}(x) + \epsilon(x, h)]$$

where $E\text{-}\lim_{h\to 0}\epsilon(x, h) = 0$. From Lemma 4.10 for each sufficiently small nonzero $h \in E$ there is a θ between 0 and 1 such that

$$\frac{(k-2)!}{h^{k-2}}\left\{f(x+h) - \sum_{n=0}^{k-1}\frac{h^n}{n!}f_{(n)}(x)\right\}$$
$$= f_{(k-2)}(x+\theta h) - f_{(k-2)}(x) - \theta h f_{(k-1)}(x).$$

Hence

$$((k-2)!/h^{k-2})\{(h^k/k!)[f_{(k)}(x) + \epsilon(x, h)]\}$$
$$= f_{(k-2)}(x+\theta h) - f_{(k-2)}(x) - \theta h f_{(k-1)}(x).$$

Thus

$$f_{(k-2)}(x+\theta h) = f_{(k-2)}(x) + \theta h f_{(k-1)}(x) + \frac{h^2}{k(k-1)}[f_{(k)}(x) + \epsilon(x, h)]$$
$$> f_{(k-2)}(x) + \theta h f_{(k-1)}(x)$$

for all sufficiently small nonzero $h \in E$. Thus, it follows by Lemma 4.16 that $f_{(k-2)}$ is convex on (c, d); hence $f_{(k-1)}$ is nondecreasing on (c, d). Choose λ between c and d. Then $f_{(k-1)}$ is bounded below on $[\lambda, d]$. Applying (ii) of the induction hypothesis to the function $f_{(k-1)}$ on the interval $[\lambda, d]$, it follows that $f_{(k-1)} = f^{(k-1)}$ on $[\lambda, d]$. Now since $f_{(k-1)}$ is nondecreasing on (λ, d) and has the Darboux property on $[\lambda, d]$ we have that $f_{(k-1)}$ is continuous and nondecreasing on $[\lambda, d]$. Similarly, since $f_{(k-1)}$ is bounded above on $[c, \lambda]$, we deduce that $f_{(k-1)}$ is continuous and nondecreasing on $[c, \lambda]$. Thus, it follows that $f_{(k-1)}$ is continuous and nondecreasing on $[c, d]$. In particular, $f_{(k-1)}$ is nondecreasing and continuous in the closure of each component of G.

To complete the proof of (i) we show $G = [a, b]$. Let $H = [a, b] - G$. From above H is a closed set having no isolated points. Since $f_{(k-1)}$ is a function of Baire class one on $[a, b]$ (see [2]), the same type of argument given in the proof of Lemma 4.20 shows H is empty. Hence $G = [a, b]$ and the proof of (i) is complete.

Consider, finally, (ii) for $k > 2$. It is no loss of generality to suppose that $f_{(k)} > 0$ on $[a, b]$. By (i), $f_{(k-1)}$ is nondecreasing on $[a, b]$ and by (ii) of the induction hypothesis $f_{(k-1)} = f^{(k-1)}$ on $[a, b]$. Thus, it follows that $f_{(1)} = f'$ on $[a, b]$. We shall prove that $(f_{(1)})_{(k-1)} = f_{(k)}$ on $[a, b]$. It will then follow by the induction hypothesis (ii) applied to $f_{(1)}$ that in $[a, b]$, $f_{(k)} = (f_{(1)})_{(k-1)} = (f_{(1)})^{(k-1)} = f^{(k)}$.

It suffices to prove that in $[a, b)$ the $(k-1)$th approximate Peano derivative of $f_{(1)}$ on the right equals $f_{(k)}$. For, applying Corollary 4.8, it will follow that in $(a, b]$ the $(k-1)$th approximate Peano derivative of $f_{(1)}$ on the left equals $f_{(k)}$. Without altering $f_{(k)}$, by adding to f a suitable polynomial of degree less than k, we may assume that $f_{(j)}(a) = 0$ for $j = 2, 3, \ldots, (k-1)$. Note, since $f^{(k-1)}(a) = 0$ and $f^{(k-1)}$ is nondecreasing on $[a, b]$, $f^{(k-1)} \geqslant 0$ on $[a, b]$. Now for each h, $0 < h < (b-a)$, there exists by the extended mean value theorem a number ξ, $a < \xi < a + h$ such that

$$f^{(2)}(a+h) = \frac{h^{k-3}}{(k-3)!} f^{(k-1)}(\xi).$$

Hence $f^{(2)} \geqslant 0$ in (a, b). Thus, $f_{(1)}$ is nondecreasing on $[a, b]$. By Lemma 4.2, $(f_{(1)})_{(k-1)}(a) = f_{(k)}(a)$. Since a may be replaced throughout by any $\alpha \in [a, b)$ the proof of the theorem is complete.

5. **The Darboux and Denjoy properties.** Neugebauer [5] proved that if g is a function of Baire class one on an interval J, then g has the Darboux property on J if and only if for each real number λ, the sets $E^{\lambda} = \{x \in J: g(x) \geqslant \lambda\}$ and $E_{\lambda} = \{x \in J: g(x) \leqslant \lambda\}$ have closed components relative to J. We thus have the following corollary to Theorem 4.1.

COROLLARY 5.1. *If $f_{(k)}$ is defined on $[a, b]$ then $f_{(k)}$ has the Darboux property on $[a, b]$.*

PROOF. Since $f_{(k)}$ is of Baire class one on $[a, b]$ (see [2]), in order to show $f_{(k)}$ has the Darboux property we need only show that the components of the sets $E^{\lambda} = \{x: f_{(k)}(x) \geqslant \lambda\}$ and $E_{\lambda} = \{x: f_{(k)}(x) \leqslant \lambda\}$ are closed for each real number λ. So suppose $f_{(k)}(x) \geqslant \lambda$ for all x in the interval (α, β). We must show that $f_{(k)}(\alpha) \geqslant \lambda$ and $f_{(k)}(\beta) \geqslant \lambda$. Since $f_{(k)}$ is bounded below on (α, β), $f_{(k)}$ is bounded below on $[\alpha, \beta]$. Thus by Theorem 4.1, $f_{(k)} = f^{(k)}$ on $[\alpha, \beta]$. Since $f^{(k)}$ has the Darboux property on $[\alpha, \beta]$, $f^{(k)}(\alpha) \geqslant \lambda$ and $f^{(k)}(\beta) \geqslant \lambda$. Hence, $f_{(k)}(\alpha) \geqslant \lambda$ and also $f_{(k)}(\beta) \geqslant \lambda$. Thus, the components of E^{λ} are closed. Similarly, the components of E_{λ} are closed. Hence, $f_{(k)}$ has the Darboux property on $[a, b]$.

In [9], Weil proved that a function g of Baire class one has the Denjoy property on an interval J if, for every subinterval L of J on which g is bounded either above or below, g restricted to L has the Denjoy property. Since an ordinary kth derivative has the Denjoy property, we also have the following corollary to Theorem 4.1.

COROLLARY 5.1. *If $f_{(k)}$ is defined on $[a, b]$, then $f_{(k)}$ has the Denjoy property on $[a, b]$.*

6. **Property Z.** To prove that $f_{(k)}$ has Property Z we first need a lemma which is a slight generalization of a lemma due to Weil [10].

LEMMA 6.1. *Suppose f is a function whose kth derivative exists and is nonnegative on the interval $[a, b]$, and let $A = \{x \in [a, b] : f^{(k)}(x) \geqslant \epsilon\}$ where ϵ is a fixed positive number. Then there exists a partition $\{a = t_0 < t_1 < \dots < t_l = b\}$ of the interval $[a, b]$ with $l \leqslant 2^k$ and such that for each $i = 1, 2, \dots, l$ with $x, y \in [t_{i-1}, t_i]$ and $x \leqslant y$*

$$|f(y) - f(x)| \geqslant (\epsilon/k!)(m(A \cap [x, y]))^k.$$

PROOF. It will be shown by induction that for each integer $j = 1, 2, \dots, k$, there is a partition of $[a, b]$,

$$\{a = t_{0,j} < t_{1,j} < \dots < t_{l(j),j} = b\},$$

with $l(j) \leqslant 2^j$ and such that for each $i = 1, 2, \dots, l(j)$ one of the following holds on $I_{i,j} = [t_{i-1,j}, t_{i,j}]$.

1(j): $f^{(k-j)} \geqslant 0$ on $I_{i,j}$ and for each $x, y \in I_{i,j}$ with $x \leqslant y$,

$$f^{(k-j)}(y) - f^{(k-j)}(x) \geqslant (\epsilon/j!)(m(A \cap [x, y]))^j.$$

2(j): $f^{(k-j)} \leqslant 0$ on $I_{i,j}$ and for each $x, y \in I_{i,j}$ with $x \leqslant y$,

$$f^{(k-j)}(y) - f^{(k-j)}(x) \geqslant (\epsilon/j!)(m(A \cap [x, y]))^j.$$

3(j): $f^{(k-j)} \leqslant 0$ on $I_{i,j}$ and for each $x, y \in I_{i,j}$ with $x \leqslant y$,

$$f^{(k-j)}(x) - f^{(k-j)}(y) \geqslant (\epsilon/j!)(m(A \cap [x, y]))^j.$$

4(j): $f^{(k-j)} \geqslant 0$ on $I_{i,j}$ and for each $x, y \in I_{i,j}$ with $x \leqslant y$

$$f^{(k-j)}(x) - f^{(k-j)}(y) \geqslant (\epsilon/j!)(m(A \cap [x, y]))^j.$$

The desired partition is then the one corresponding to $j = k$ and the desired inequality is obtained by taking absolute values, where, of course $f^{(0)} = f$.

If the conditions 1(j)–4(j) above are used in place of Weil's conditions 1(j)–4(j) in [10], then the reader may complete the proof by making the rather obvious changes in Weil's proof in [10].

THEOREM 6.2. *If f has a kth approximate Peano derivative $f_{(k)}$ everywhere on $[a, b]$ then $f_{(k)}$ has Property Z on $[a, b]$.*

PROOF. Let x be contained in $[a, b]$ and $\epsilon > 0$. It suffices to show that if given an $\eta > 0$ there exists a $\delta > 0$ such that if the closed interval $[\alpha, \beta]$ is contained in $(x - \delta, x + \delta) \cap [a, b]$, $x \notin [\alpha, \beta]$ and $f_{(k)}(y) \geqslant f_{(k)}(x)$ for each $y \in [\alpha, \beta]$ or $f_{(k)}(y) \leqslant f_{(k)}(x)$ for each $y \in [\alpha, \beta]$ then

$$\frac{m\{y \in [\alpha, \beta] : |f_{(k)}(y) - f_{(k)}(x)| \geqslant \epsilon\}}{(\beta - \alpha) + \operatorname{dist}(x, [\alpha, \beta])} < \eta. \tag{6.3}$$

Let $\eta > 0$ be given and set

$$g(y) = f(y) - \sum_{n=0}^{k} \frac{(y - x)^n}{n!} f_{(n)}(x).$$

Then $g_{(k)}(y)$ exists for each $y \in [a, b]$ and furthermore

$$g_{(k)}(y) = f_{(k)}(y) - f_{(k)}(x).$$

From the existence of $f_{(k)}$, there exists a $\delta > 0$ and a measurable set $E \subset [a, b]$ such that x is a point of density of E, and so that

$$|g(y)| \leqslant \frac{\epsilon(\eta/2)^k}{k! \cdot 2^{k(k+1)}} \cdot |y - x|^k \tag{6.4}$$

for $|y - x| < \delta$ and $y \in E$,

$$m(J \cap E^c) \leqslant m(J) \cdot \eta/2 \tag{6.5}$$

for J an interval contained in $(x - \delta, x + \delta) \cap [a, b]$ and $x \in J$, where $E^c = [a, b] - E$.

Let $[\alpha, \beta]$ be a closed interval contained in $(x - \delta, x + \delta) \cap [a, b]$ such that $x \notin [\alpha, \beta]$. First assume that $f_{(k)}(y) \geqslant f_{(k)}(x)$ for each $y \in [\alpha, \beta]$. By Theorem 4.1, $f_{(k)} = f^{(k)}$ on $[\alpha, \beta]$. Applying Lemma 6.1 to the function g, which satisfies $g^{(k)}(y) = f_{(k)}(y) - f_{(k)}(x)$ for each $y \in [\alpha, \beta]$, there exists a partition of $[\alpha, \beta]$, $\{\alpha = t_0 < t_1 < \ldots < t_l = \beta\}$, with $l \leqslant 2^k$ such that for each $i = 1, 2, \ldots, l$ and each $s, w \in [t_{i-1}, t_i]$ with $s \leqslant w$,

$$|g(w) - g(s)| \geqslant (\epsilon/k!)(m(A \cap [s, w]))^k \tag{6.6}$$

where $A = \{y \in [\alpha, \beta] : |g^{(k)}(y)| = |f_{(k)}(y) - f_{(k)}(x)| \geqslant \epsilon\}$. If $f_{(k)}(y) \leqslant f_{(k)}(x)$ for each $y \in [\alpha, \beta]$, then consider $-g$ and apply Lemma 6.1 to obtain precisely the same inequality (6.6).

We first obtain an estimate for $m(A \cap E)$. For this purpose assume $[t_{i-1}, t_i] \cap E \neq \varnothing$. Let $t_{i-1} \leqslant t_i' \leqslant t_i'' \leqslant t_i$ with $t_i', t_i'' \in E$. Then by (6.6) and (6.4)

$$\begin{aligned} m(A \cap [t_i', t_i'']) &\leqslant (k!/\epsilon)^{1/k}|g(t_i'') - g(t_i')|^{1/k} \\ &\leqslant (k!/\epsilon)^{1/k}(|g(t_i'')|^{1/k} + |g(t_i')|^{1/k}) \\ &\leqslant \left(\frac{k!}{\epsilon}\right)^{1/k}\left(\frac{\epsilon(\eta/2)^k}{k!\cdot 2^{k(k+1)}}\right)^{1/k}(|t_i''-x| + |t_i'-x|) \\ &\leqslant (\eta/[2\cdot 2^k])[\mathrm{dist}(x, [\alpha, \beta]) + (\alpha - \beta)] \\ &\leqslant (\eta/2l)[\mathrm{dist}(x, [\alpha, \beta]) + (\alpha - \beta)]. \end{aligned}$$

If

$$s_i' = \inf\{t_i' : t_i' \in [t_{i-1}, t_i] \cap E\}$$

and

$$s_i'' = \sup\{t_i'' : t_i'' \in [t_{i-1}, t_i] \cap E\}$$

then it follows from the above inequality that

$$\begin{aligned} m(A \cap E \cap [t_{i-1}, t_i]) &= m(A \cap E \cap [s_i', s_i'']) \\ &\leqslant m(A \cap [s_i', s_i'']) \leqslant (\eta/2l)[\mathrm{dist}(x, [\alpha, \beta]) + (\beta - \alpha)]. \end{aligned}$$

Clearly the same estimate holds if $[t_{i-1}, t_i] \cap E = \varnothing$. Hence

$$\begin{aligned} m(A \cap E) &= m\left(A \cap E \cap \left(\bigcup_{i=1}^{l} [t_{i-1}, t_i]\right)\right) \\ &= \sum_{i=1}^{l} m(A \cap E \cap [t_{i-1}, t_i]) \\ &\leqslant \sum_{i=1}^{l} \frac{\eta}{2l}[\mathrm{dist}(x, [\alpha, \beta]) + (\beta - \alpha)] \\ &\leqslant \frac{\eta}{2}[\mathrm{dist}(x, [\alpha, \beta]) + (\beta - \alpha)]. \end{aligned} \tag{6.7}$$

Secondly, we obtain an estimate of $m(A \cap E^c)$. Let J be the smallest closed interval in $[a, b]$ containing both x and $[\alpha, \beta]$. Using (6.5) we have the following estimate

$$m(A \cap E^c) \leqslant m(J \cap E^c) \leqslant (\eta/2)\cdot m(J) \leqslant (\eta/2)[\operatorname{dist}(x, [\alpha, \beta]) + (\beta - \alpha)]. \tag{6.8}$$

Therefore by (6.7) and (6.8)

$$m(A) = m(A \cap E) + m(A \cap E^c) \leqslant [\operatorname{dist}(x, [\alpha, \beta]) + (\beta - \alpha)] \cdot \eta$$

and (6.3) holds. Thus, $f_{(k)}$ has Property Z on $[a, b]$ and the proof is complete.

Property Z was first introduced by Weil [10]. He further showed in the same paper that if a function g has the Darboux property and Property Z then g also has the Zahorski property. (An example of a function having the Darboux property and the Zahorski property but not Property Z can be found in [10].) Hence, in the class of functions having the Darboux property, Property Z is strictly stronger than the Zahorski property.

Thus by Corollary 5.1 we have the following corollary to the last theorem.

COROLLARY 6.9. *If $f_{(k)}$ is defined on $[a, b]$, then $f_{(k)}$ has the Zahorski property on $[a, b]$.*

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